

# Covariant Extended Phase Space for Fields on Curved Background

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## Abstract

It is shown that the nature of physical time requires the extended phase space in mechanics to have a bundle structure with time as the 1-dimensional base manifold and the phase space as the fiber. Phase trajectories are sections whose tangent vectors annihilate the 2-form  $-d\Theta$  where  $\Theta$  is the Poincare-Cartan 1-form  $\Theta = pdq - Hdt$ . This bundle picture of the extended phase space is then applied to fields in a covariant, ‘directly’ Hamiltonian formalism without requiring a Lagrangian as a starting point. In this formalism the base manifold is the four-dimensional space-time with a Riemannian metric. The canonical momenta are differential 1-forms for each field degree of freedom. The Poincare-Cartan 4-form has the general structure  $\Theta = (*p) \wedge d\phi - H$  where  $*$  is the Hodge star operator of the Riemannian metric. Allowed field configurations are sections of the bundle such that the 4-dimensional tangent spaces to these sections, annihilate the 5-form  $-d\Theta$ . Noether currents are calculated for symmetry fields and a new bracket analogous to the Peierls bracket is defined on the extended phase space in place of the Poisson bracket.

## 1 Introduction

Formulating a dynamical problem in classical mechanics is always ‘directly Hamiltonian’. Given a configuration space  $Q$ , its cotangent bundle  $T^*(Q)$  possesses a unique fundamental 1-form  $p_i dq^i$  (summation

over  $i$  implied). All one needs is a Hamiltonian function  $H(q, p)$  and the dynamics is fixed by the Poincare-Cartan (PC) form  $p_i dq^i - H dt$ .

But a relativistic *field* theory, it is thought, must always begin with a Lagrangian because relativistic invariance is spoiled by the momenta which are related to time derivative of fields. A great deal of effort is required to re-establish relativistic invariance [1]. The subject of this paper is a ‘directly Hamiltonian’, relativistically covariant field theory which offers additional insights.

The first step in this direction is to treat all the four derivatives of field  $\partial_\mu \phi$  on the same footing. We can do that by writing  $d\phi = \partial_\mu \phi dx^\mu$ . This is analogous to  $dq$  of mechanics. Since for the simplest cases  $p$  is proportional to  $\dot{q}$  in mechanics, we guess that we should *define the canonical momenta for scalar fields to be differential 1-forms* like  $d\phi$ .

The all-important quantity *action* is simply the integral of the PC-form  $\int (p_i dq^i - H dt)$ . In field theory it is an integral over a four-dimensional domain. Thus the PC-form for fields should be a differential 4-form. To write  $p \wedge d\phi$  will give us only a 2-form. But if we use the metric of space-time, we can convert a 1-form  $p$  into a 3-form via the Hodge star operator. So we suggest the PC-form to be of the form  $(*p) \wedge d\phi - H$  where  $H$ , the ‘Hamiltonian’ is a suitable 4-form made solely out of  $\phi$  and  $p$  (or  $*p$ ). No  $d\phi$  or  $dp$  should be used in  $H$ . The 4-form  $H$  should not be confused with the energy density which is a 3-form having constant integral on space-like surfaces in static space-times.

The purpose of this paper is to outline a new formulation of fields in a curved background. The main features of the formalism are as follows :

1. It is a totally covariant, directly Hamiltonian formalism in the extended phase space (called EPS hereafter) which includes space-time as well as the fields and momenta. There is no Lagrangian defined. Instead a covariant differential 4-form  $H$  determines the dynamics.
2. EPS has the structure of a bundle with 4-dimensional space-time as the base. The typical fiber is the phase space of field variables and canonical momenta. In this we are prompted by classical mechanics where the bundle structure is forced by unidirectional nature of physical time [2].
3. The field variables are vector space valued differential forms. The canonical momenta are differential forms of a degree higher.

4. The central objects of the theory are the Poincare-Cartan (PC) differential 4-form  $\Theta$  (analogous to the 1-form  $pdq - Hdt$  of mechanics) and its derivative 5-form  $\Xi = -d\Theta$ . (The negative sign is just a convention.)
5. The quantity  $\int \Theta$  over a 4-dimensional surface in the EPS is the action. we limit ourselves to 4-dimensional surfaces which are *sections*, that is, mappings from the space-time base into the bundle respecting the fiber structure.
6. Stationarity of action implies that the allowed field configurations are those sections or surfaces whose tangent spaces annihilate the 5-form  $\Xi = -d\Theta$ . This condition is precisely equivalent to the field equations. We call these sections as ‘solution surfaces’. This is completely analogous to Hamiltonian mechanics where phase trajectories annihilate the 2-form  $-d(p_i dq^i - Hdt)$ .
7. Observables of the theory are generally of the form  $\int A$  where  $A$  is a 4-form defined on the EPS. By suitably ‘smearing’ the 4-form  $A$  with functions of compact support we can also include the local observables of field theory.
8. Fields and canonical momenta are forms of different degrees. There cannot be a Poisson bracket in the usual sense. Instead, a covariant bracket defined by Peierls long ago is the natural bracket in our formalism. Peierls bracket is a generalization of the Poisson bracket in the sense that equal-time Peierls brackets coincide with Poisson brackets when time is singled out and time derivative of the field related to canonical momentum as in the usual ‘3+1’ formulation of field theory.
9. The Riemann metric of the base manifold plays a crucial role in the formalism through the Hodge star operator which converts  $r$ -forms into  $(4 - r)$ -forms.
10. The formalism limits the kind of fields that can exist so far as their differential-geometric nature in space-time is concerned. Because of the star duality, the fields can only be 0, 1 or 2-forms. However, there is no restriction on the internal space. The theory can cover all known matter and gauge fields in arbitrary background space-time.
11. Symmetries, including gauge symmetries are mappings on the EPS that leave the PC form invariant. For infinitesimal symmetry mappings generated by a vector field  $Y$  on EPS, the 3-form

$i(Y)\Theta$  integrated on a closed 3-surface of the 4-dimensional solution surface is zero. This is simply the statement of Noether's theorem.

In the next section we review the extended phase space in mechanics. The requirement that physical time always moves forward reveals a bundle structure of the EPS. The role of PC-form and its derivative 2-form  $\Xi = -d\Theta$  is examined, particularly the factorization properties of  $\Xi$ . In section 3 we begin to construct the extended phase space and PC-form for fields for scalar fields. In section 4 Noether currents are defined and calculated. Section 5 explains the concept of observables and the Peierls brackets in this formalism. The last section is devoted to briefly enumerating ideas on variational principle for fields which have inspired the present formalism.

## 2 Bundle structure in Classical Mechanics

### 2.1 Extended Phase Space

There are two ways to look at the evolution of a classical mechanical system in the phase space.

One, a phase space, at least locally, has a fundamental form  $\theta = p_i dq^i$  defined on it. Its exterior derivative (with a conventional negative sign),  $-d\theta = dq^i \wedge dp_i$ , is a non-degenerate 2-form. In other words, its components are an antisymmetric matrix with determinant non-zero. This 2-form can be used to convert vector fields on the phase space into differential 1-forms by contraction. The inverse to this mapping associates a vector field to a 1-form. For any function  $f(q, p)$  on the phase space the vector field associated with the 1-form  $df$  is called the Hamiltonian vector field of  $f$ , and it is given by

$$X_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}.$$

Integral curves of the vector field  $X_H$  of the Hamiltonian function  $H(q, p)$  are the evolution trajectories.

Equivalently, one can formulate dynamics in the *extended* phase space [3]. Here, time is taken as an additional dimension adjoined to the phase space. The fundamental form  $\theta$  is augmented to the Poincare-Cartan 1-form (PC form)  $\Theta \equiv p_i dq^i - H dt$  where  $H$  is the

Hamiltonian function. A curve  $C$  in this  $(2n + 1)$ -dimensional space is called a *characteristic curve* of the 2-form  $\Xi \equiv -d\Theta$  if the tangent vectors  $X$  to the curve  $C$  “annihilate  $\Xi$ ” : that is, as a 1-form

$$i(X)\Xi = 0,$$

where  $i(X)$  denotes the interior product of the vector field  $X$  with what follows after it. The form  $\Xi$  will play a central role in our formulation. The class of characteristic curves (or 1-dimensional sub-manifolds) provides solutions to the dynamical problem. Let  $C : s \rightarrow (t(s), q(s), p(s))$  be a curve and  $H(q, p)$  the Hamiltonian function. The tangent vector to the curve is

$$X = \frac{d}{ds} = \frac{\partial t}{\partial s} \frac{\partial}{\partial t} + \frac{dq^i}{ds} \frac{\partial}{\partial q^i} + \frac{dp_i}{ds} \frac{\partial}{\partial p_i},$$

and the derivative of the PC form is

$$\begin{aligned} \Xi = -d\Theta &= dq^i \wedge dp_i + dH \wedge dt \\ &= dq^i \wedge dp_i + \frac{\partial H}{\partial q^i} dq^i \wedge dt + \frac{\partial H}{\partial p_i} dp_i \wedge dt. \end{aligned} \quad (1)$$

The interior product of  $X$  with 2-form  $\Xi$  is the 1-form

$$\begin{aligned} i(X)\Xi &= -\frac{\partial H}{\partial q^i} \frac{dt}{ds} dq^i - \frac{\partial H}{\partial p_i} \frac{dt}{ds} dp_i + \frac{dq^i}{ds} dp_i \\ &+ \frac{\partial H}{\partial q^i} \frac{dq^i}{ds} dt - \frac{dp_i}{ds} dq^i + \frac{\partial H}{\partial p_i} \frac{dp_i}{ds} dt \\ &= \left( \frac{\partial H}{\partial q^i} \frac{dq^i}{ds} + \frac{\partial H}{\partial p_i} \frac{dp_i}{ds} \right) dt - \left( \frac{\partial H}{\partial q^i} \frac{dt}{ds} + \frac{dp_i}{ds} \right) dq^i \\ &+ \left( -\frac{\partial H}{\partial p_i} \frac{dt}{ds} + \frac{dq^i}{ds} \right) dp_i. \end{aligned}$$

Thus  $X$  annihilates  $\Xi$  if

$$\frac{dH}{ds} = 0, \quad (2)$$

$$\frac{dq^i}{ds} = \frac{\partial H}{\partial p_i} \frac{dt}{ds}, \quad (3)$$

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial q^i} \frac{dt}{ds}. \quad (4)$$

These are equivalent to the canonical equations of motion

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

provided  $(dt/ds) \neq 0$ .

The seemingly obvious requirement  $dt/ds \neq 0$  is quite deep in fact. Physically,  $(dt/ds) > 0$  (for example) means that *time never stops and always moves forward* with parameter  $s$ . Mathematically,  $(dt/ds) > 0$  suggests a bundle picture as follows.

$dt/ds$  is the push-forward, under the projection  $(t, q, p) \rightarrow t$ , of the tangent vector to the 1-dimensional sub-manifold  $s \rightarrow (t(s), q(s), p(s))$  into the tangent vector to the projected curve  $s \rightarrow t(s)$ .

Therefore if the physical time  $t$  is chosen as the one-dimensional *base* manifold with the phase space with coordinates  $(q, p)$  as fiber, and  $\pi : (t, q, p) \rightarrow t$  the projection then  $C : s \rightarrow (t(s), q(s), p(s))$  is a “projectable” sub-manifold, that is, a manifold whose (non-zero) tangent vectors map to non-zero vectors on the base by the push-forward map  $\pi_*$ . We can equivalently say that the allowed trajectories in the extended phase space are “sections”  $t \rightarrow (t, q(t), p(t))$  from the one-dimensional base space into the bundle which are, moreover, characteristics of  $\Xi$ . In other words, the tangent spaces of the sub-manifold annihilate  $\Xi$ .

It is important to realize that the “arrow of time” (either  $dt/ds > 0$  for all  $s$  or  $dt/ds < 0$  for all  $s$ ) is a consequence of the continuous nature of the section. We shall see that causality, or the unidirectional nature of time for cause and effect, remains an important ingredient for fields in extended phase space formalism as well.

## 2.2 Variational Principle

The characteristic curves of the two-form  $\Xi = -d\Theta$  are related to the variational principle in the extended phase-space. Let  $C : s \rightarrow (t(s), q(s), p(s))$  be as above with  $s \in (s_1, s_2)$ . The *action*, defined as the integral of the PC form on this curve is the quantity

$$A(C) = \int_C \Theta = \int_C (p_i dq^i - H dt).$$

Let  $Y$  be a vector field in the extended phase space representing infinitesimal variations.  $Y$  need only be defined in the neighborhood of the (image of the) curve  $C$ . The action  $A(C)$  is stationary if its Lie

derivative  $\mathcal{L}_Y A(C) = \int \mathcal{L}_Y \Theta$  with respect to  $Y$  is zero. Using the formula  $\mathcal{L}_Y = i(Y) \circ d + d \circ i(Y)$  which is true when acting on differential forms, we calculate

$$\mathcal{L}_Y(A(C)) = - \int i(Y) \Xi + [i(Y)(p_i dq^i - H dt)] \Big|_{s_1}^{s_2},$$

where the second term, which comes from the exact differential  $d \circ i(Y) \Theta$  is zero if variation of  $q^i$ 's and  $t$  is zero on the boundary points at  $s = s_1, s_2$ . There is no restriction on variation of  $p_i$ 's however. The first term when evaluated along  $C$  will be the integral of  $-(i(Y) \Xi)(X)$  where  $X$  is the tangent vector along  $C$ . Thus,

*for variation field  $Y$ , which is arbitrary except for the boundary conditions noted above, action is stationary if and only if  $i(Y) \Xi$  evaluated on the tangent to the curve  $C$  is zero at all points on  $C$ .*

## 2.3 Factorization Property

In order to generalize to field theory we demonstrate a useful factorization property of  $\Xi$ . Refer to equation 1 above and write

$$\Xi = \left( dq^i - \frac{\partial H}{\partial p_i} dt \right) \wedge \left( dp_i + \frac{\partial H}{\partial q^i} dt \right) \quad (5)$$

where a superfluous term proportional to  $dt \wedge dt = 0$  has been introduced to obtain the factorization. The variational principle says that for arbitray variational field  $Y$ , the 1-form  $i(Y) \Xi$  must vanish on the proposed phase trajectory. If we choose  $Y = \partial / \partial q^1$ , for example, then

$$i(Y) \Xi = \left( dp_1 + \frac{\partial H}{\partial q^1} dt \right).$$

Evaluated on the tangent vector to the trajectory  $t \rightarrow (t, q^i = F^i(t), p_i = G_i(t))$  it gives

$$\frac{dG_1}{dt} = - \frac{\partial H}{\partial q^1} \Big|_{q=F, p=G}.$$

By choosing  $Y$  in different directions of the phase space we get all the Hamilton's equations.

### 3 Scalar fields

Usually a field system is said to involve infinitely many degrees of freedom. According to this traditional view each value  $\phi(\mathbf{x}, t)$  for space points  $\mathbf{x}$  on a plane of constant time  $t$  is a separate degree of freedom for a scalar field. This is the usual ‘3+1’ Hamiltonian point of view. See Chernoff and Marsden[4] for a rigorous account of Hamiltonian systems of infinitely many degrees of freedom. The global-geometric view taken by our approach is to regard the entire field configuration over all space-time as a 4-dimensional surface ‘above’ space-time in the extended phase-space. The infinitely many degrees of freedom appear only when time is chosen as a special parameter of evolution.

We now generalize the idea to the extended phase space for fields. The bundle picture remains, except that the 1-dimensional base space of time is now replaced by the 4-dimensional space-time. We consider the scalar field in arbitrary curved back ground to illustrate the idea.

In field theory, the field  $\phi$  is the configuration variable analogous to  $q$ . Time and space are four “time” variables  $t^\mu, \mu = 0, 1, 2, 3$ . The PC-form for fields is a differential *four*-form whose space-time integral is the action. As discussed in the Introduction, for a scalar field  $\phi$  its canonical momentum is a differential 1-form  $p = p_\mu dx^\mu$  and the PC 4-form looks like :

$$\Theta = (*p) \wedge d\phi - H \quad (6)$$

where the ‘Hamiltonian’  $H$  is a differential 4-form made from  $\phi$  and  $p$  only. We choose it to be

$$\begin{aligned} H &= \frac{1}{2}(*p) \wedge p + \frac{1}{2}m^2\phi^2(*1) \\ &= \left( -\frac{1}{2}\langle p, p \rangle + \frac{1}{2}m^2\phi^2 \right) (*1). \end{aligned} \quad (7)$$

We have used the definition of the star operator relating it to the inner product determined by  $g_{\mu\nu}$  because

$$*(dt^\mu) \wedge (dt^\nu) = -g^{\mu\nu}(*1). \quad (8)$$

Our notation is the same as Sharan[5] or Choquet-Bruhat and DeWitt-Morette[6].

For our case  $\Xi$  is a differential 5-form and it can be calculated easily. Using

$$d(*p \wedge p) = d(*p) \wedge p + *p \wedge (dp) = 2d(*p) \wedge p$$



we get

$$dH = (d * p) \wedge p + m^2 \phi d\phi \wedge (*1).$$

Substituting in  $\Xi = -d\Theta$  we see that it factorizes :

$$\Xi = (d * p - m^2 \phi(*1)) \wedge (p - d\phi), \quad (9)$$

where we use the fact that the 5-form  $(*1) \wedge p$  in four variables  $t$  is zero because there are five factors of  $dt$ 's. This is completely analogous to the factorization of  $\Xi$  in mechanics.

We now use our variational principle. Take  $Y = \partial/\partial\phi$ , and the interior product  $i(Y)\Xi$  is the 4-form

$$i\left(\frac{\partial}{\partial\phi}\right)\Xi = -(d * p - m^2 \phi(*1)).$$

When evaluated on the section  $t \rightarrow \phi(t), p_\mu(t)dt^\mu$  it gives

$$-d * [p_\mu(t)dt^\mu] + m^2 \phi(*1) = 0.$$

If  $Y$  is chosen to be  $\partial/\partial p_\nu$  then because  $d*p = d*(p_\mu dt^\mu) = dp_\mu \wedge *(dt^\mu)$  we have

$$\begin{aligned} i\left(\frac{\partial}{\partial p_\nu}\right)\Xi &= *(dt^\nu) \wedge \left(p_\mu(t) - \frac{\partial\phi}{\partial t^\mu}\right) dt^\mu \\ &= -g^{\nu\mu} \left(p_\mu(t) - \frac{\partial\phi}{\partial t^\mu}\right) \\ &= 0. \end{aligned}$$

This gives us four equations

$$p_\mu - \frac{\partial\phi}{\partial t^\mu} = 0. \quad (10)$$

Thus (now treating  $\phi$  and  $p$  as functions of  $t$ ) we get the Hamiltonian equations

$$p = d\phi, \quad d * p - m^2 \phi(*1) = 0. \quad (11)$$

The second of these equations is the Klein-Gordon equation for  $\phi$  on curved background when the first is substituted in it because

$$\begin{aligned} d * d\phi &= \partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi)dt^0 \wedge \dots \wedge dt^3 \\ &= \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi)(*1) \end{aligned} \quad (12)$$

## 4 Noether Currents

Let  $D$  be a domain in the 4-dimensional base of space-time. Let  $\sigma$  be the mapping from the base into the bundle. Then action for this section is

$$A(\sigma) = \int_{\sigma(D)} \Theta. \quad (13)$$

The variational principle for fields (in complete analogy to mechanics) can be written as

$$\begin{aligned} \delta_Y A(\sigma) &= \int_{\sigma(D)} \mathcal{L}_Y \Theta \\ &= \int_{\sigma(D)} d[i_Y \Theta] - \int_{\sigma(D)} i_Y \Xi \\ &= \oint_{\partial\sigma(D)} i_Y \Theta - \int_{\sigma(D)} i_Y \Xi \end{aligned} \quad (14)$$

where  $Y$  is the variational field in the EPS. If  $\phi$  is kept fixed on the boundary then  $Y$  has zero component in the direction of  $\phi$  and  $i(Y)\Xi = 0$ . Consequently, the first term is zero and the equations of motion are obtained from  $\delta_Y A(\sigma) = 0$  as the condition that the 4-form  $i(Y)\Xi$  should vanish when evaluated on the tangent vectors to the 4-dimensional section or solution-surface. This is the criterion we have used.

On the other hand if we already have a solution surface and  $Y$  is arbitrary then the second term is zero and the variation is

$$\int_{\sigma(D)} \mathcal{L}_Y \Theta = \oint_{\partial\sigma(D)} i_Y \Theta. \quad (15)$$

A symmetry transformation is given by a field  $Y$  such that  $\mathcal{L}_Y \Theta = 0$ . Then for such symmetry transformations

$$\oint_{\partial\sigma(D)} i_Y \Theta = 0. \quad (16)$$

The 3-form  $i_Y \Theta$  is called the Noether current and the statement above is the conservation law. If the boundary  $\partial(\sigma(D))$  is taken to be of the shape of two space-like surfaces joined together at spatial infinity. Then the surface integral involves the  $dt^1 \wedge dt^2 \wedge dt^3$  component of the Noether 3-form. This is the more usual statement of the Noether theorem.

Usually, the symmetry fields satisfy the stronger conditions  $L_Y(*p \wedge d\phi) = 0$  and  $L_Y H = 0$  separately.

As an example, if

$$Y = v^\mu(t) \frac{\partial}{\partial t^\mu}$$

then the Noether current 3-form is given by,

$$i_Y \Theta = \left[ \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \left( g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + m^2 \phi^2 \right) \right] v^\mu * (dt^\nu) \quad (17)$$

While calculating these currents one must not assume the fields  $\phi$  and  $p$  to be on the solution surface to begin with. Only after taking interior product (that is contracting) with  $Y$  can the field be evaluated at the solution surface. The details of this not entirely trivial calculation are given in Appendix A. For Minkowski space-time admitting translational symmetry the  $dt^1 \wedge dt^2 \wedge dt^3$  components of conserved quantities (energy and momentum densities) are the familiar expressions given below.

$Y$	$v^\mu$	Coefficient of $-dt^1 \wedge dt^2 \wedge dt^3$ in $i_Y \Theta$
$\partial/\partial t^0$	$(1, 0, 0, 0)$	$(1/2)[(\phi_{,0})^2 + (\nabla\phi)^2 + m^2\phi^2]$
$\partial/\partial t^1$	$(0, 1, 0, 0)$	$\phi_{,1}\phi_{,0}$

## 5 Observables and Peierls bracket

Our formalism treats coordinate  $\phi$  and its canonical momentum  $p$  respectively as 0- and 1-forms. In classical mechanics they seem to be quantities of the same type because in one-dimensional base manifold representing time, 0-forms and 1-forms are both 1-dimensional spaces. This situation changes for field theory in four dimensions. There 0- and 1-forms are respectively spaces of one and four dimensions.

We define observables of our theory to be integrated quantities over a four dimensional sub-manifold of the EPS. Quantities like action are a good example. A typical observable is determined by a 4-form  $A = \int \alpha$ . The support of  $\alpha$ , that is set over which it has non-zero values could be suitably restricted to allow for local quantities as observables. For example, the scalar field  $\phi$  is related to the observable  $\int \phi j(*1)$

where  $j(t)$  is a scalar ‘switching function’ which is non-zero in a small space-time region. For simplicity we would call both the integrated as well as the non-integrated quantity by the same name ‘observable’, and it leads to no confusion.

The Peierls bracket [7], promoted extensively by DeWitt [8], is the natural bracket-like quantity in this formalism. When the Hamiltonian 4-form  $H$  is perturbed by observable  $\lambda B$  (where  $\lambda$  is an infinitesimal parameter) the solution manifold shifts, and, after taking causality into account, the difference between the two solutions at different points in the limit of  $\lambda \rightarrow 0$  determines a ‘vertical’ vector field  $X_B$ . This field changes all other observables. The change in an observable  $A$  is equal to the Lie derivative  $D_B A \equiv L_{X_B} A$  of  $A$  with respect to  $X_B$ . Switching the roles of  $B$  and  $A$  we can calculate  $D_A B$ . The Peierls bracket  $[A, B]$  is defined as the difference  $D_B A - D_A B$ .

For illustration we outline the calculation of the Peierls bracket for the scalar field with itself in Minkowski space. The observable in question is the integrated 4-form

$$B = \int \beta = \int \phi j(*1)$$

where  $j$  is a switching function in space-time with which the field  $\phi$  is ‘smeared’. The Hamiltonian is changed to  $H + \lambda B$  and the solution manifold given by  $t \rightarrow \phi = F_0(t), p_\nu = F_{0,\nu}$  gets modified to a solution manifold which is determined by the 5-form

$$\begin{aligned} \Omega_B &= -d(*p) \wedge d\phi + dH + \lambda d\phi j(*1) \\ &= [d(*p) - m^2 \phi(*1) - \lambda j(*1)] \wedge [p - d\phi]. \end{aligned}$$

No derivative of  $j$  appears because that would involve five factors of  $dt$ ’s and there can be only four such factors in a wedge product. The equations for a solution  $t \rightarrow \phi = F(t), p_\nu = G_\nu$  become

$$G_\nu = F_{,\nu}, \quad (\partial^\mu \partial_\mu - m^2)F = \lambda j.$$

The modification caused by  $\lambda B$  as  $\lambda \rightarrow 0$  to the solution  $F_0$  is given by the retarded solution to the inhomogeneous Klein-Gordon equation,

$$F(t) = F_0(t) + \lambda K(t), \quad G_\nu = F_{,\nu}$$

where

$$K(t) = \int G_R(t-s) j(s) d^4 s.$$

The retarded and advanced Green's functions  $G_R(t), G_A(t)$  are the unique solutions

$$G_{R,A}(t) = \frac{1}{(2\pi)^4} \int d^4k \frac{\exp(-ik^0 t^0 + i\mathbf{k} \cdot \mathbf{t})}{(k^0 \pm i\epsilon)^2 - \mathbf{k}^2 - m^2}$$

of

$$(\partial^\mu \partial_\mu - m^2)G_R(t) = \delta^4(t)$$

with the boundary condition that  $G_R(t)$  is non-zero only in the forward light-cone and  $G_A(t)$  in the backward light-cone.

Thus the vertical field is determined to be ( $\lambda \rightarrow 0$  can be factored out to give the tangent vector field)

$$Y_B = K(t) \frac{\partial}{\partial \phi} + K_{,\nu} \frac{\partial}{\partial p_\nu}$$

Consider the observable

$$A = \int \alpha = \int \phi k(*1)$$

where  $k(t)$  is another switching function. The change in  $A$  due to  $B$  is given by  $D_B A = L_{Y_B}(A)$ . Now,

$$\begin{aligned} L_{Y_B}(A) &= \int [i_Y(d\phi k(*1)) + d(\phi k i(Y)(*1))] \\ &= \int k K(*1), \end{aligned}$$

because  $i(Y_B)(*1) = 0$ . Thus

$$\begin{aligned} D_B A &= \int d^4t k(t) K(t) \\ &= \int \int d^4t d^4s k(t) G_R(t-s) j(s) \end{aligned}$$

Reversing the role of  $B$  and  $A$  we get the Peierls bracket

$$[A, B] = D_B A - D_A B = \int \int d^4t d^4s k(t) \Delta(t-s) j(s)$$

where  $\Delta$  is the Pauli-Jordan function  $\Delta = G_R - G_A$ . This is equivalent to the commutator

$$[\phi(t), \phi(s)] = \Delta(t-s)$$

when  $k$  and  $j$  are Dirac deltas with support at  $t$  and  $s$  respectively.

The Peierls bracket for the field  $\phi$  and momentum  $p$  can be calculated by considering the observable

$$C = \lambda(*p) \wedge l = -\lambda p_\mu l^\mu(*1)$$

where in this case we must employ a 1-form switching function  $l$  to smear the momentum. The 5-form is

$$\Omega_C = [d(*p) - m^2\phi(*1)] \wedge [p + \lambda l - d\phi].$$

The relevant equation for the modified solution is

$$(\partial^\mu \partial_\mu - m^2)F = \lambda \partial^\mu l_\mu$$

because  $d(*p)$  becomes  $d(*(\partial\phi - l)) = \partial^\mu \partial_\mu \phi - \partial^\mu l_\mu$ . The change in  $B$  is

$$D_C B = \int \int d^4 t d^4 s j(t) G_R(t-s) (\partial^\mu l_\mu)(s).$$

On the other hand we have already calculated the vertical field for  $B$  which gives

$$\begin{aligned} D_B C &= - \int K_{,\mu} l^\mu(*1) \\ &= - \int \int d^4 t d^4 s l^\mu(t) \partial_{t^\mu} G_R(t-s) j(s) \\ &= \int \int d^4 t d^4 s (\partial_\mu l^\mu)(t) G_R(t-s) j(s) \\ &= \int \int d^4 t d^4 s j(t) G_A(t-s) (\partial_\mu l^\mu)(s) \end{aligned}$$

after integrating by parts in the third step. Therefore,

$$[B, C] = \int \int d^4 t d^4 s j(t) \Delta(t-s) (\partial_\mu l^\mu)(s)$$

which, for  $j(t) = \delta^4(t)$  and  $l_\mu = (1, 0, 0, 0)\delta^4(s)$  gives the equal-time  $(t^0 - s^0)$  canonical Poisson bracket of the “3+1” version of field theory

$$[\phi(t, \mathbf{t}), p_0(t, \mathbf{s})] = \delta(\mathbf{t} - \mathbf{s})$$

because

$$\partial_0 \Delta(t) = -\delta^3(\mathbf{t}).$$

## 6 Discussion and conclusion

The mathematical formalism of this paper is similar to the “multi-symplectic” Lagrangian approach to field theory in the works of Le Page, as reviewed and developed by Kastrup[9], the De Donder-Weyl[10] approach of Kanatchikov[11] and the covariant Hamiltonian-Jacobi formalism of Rovelli[12]. Recent contributions to multisymplectic formalism are by Gotay and collaborators[13]. Our approach is different from these because we use the background space-time metric in an essential way through the Hodge star operator. Also, we treat the space-time degrees of freedom  $t^\mu$  (which specify the base) very differently from the field or momentum degrees of freedom which are in the fibre above the base. We require the PC-form to be a 4-form whose first term is linear in  $d\phi$  to imitate  $pdq$  term and the second term is a 4-form  $-H$  proportional to volume form  $(*1)$ . If, for instance, there are two fields  $\phi_1$  and  $\phi_2$ , a 4-form involving a factor  $d\phi_1 \wedge d\phi_2$  is possible in principle but that does not seem to be allowed in the formalism for matter fields. Similarly other ‘non-canonical’ expressions are possible in place of the standard  $pdq - Hdt$  like expression. For gravity, the Einstein-Hilbert PC form does seem to have a non-standard expression as we shall see in a later paper. But gravity is a special case anyway. For gravity the ‘internal’ degrees of freedom in the fibre related to arbitrary choice of local inertial frames and space-time bases which define the transformation of all field and momenta differential forms happen to coincide.

It is natural and tempting to put our formalism in the topologically non-trivial situations but we avoid that in this first of a series of papers and limit to clarifying the physical concepts. Thus we assume the bundle to be a cartesian product of space-time and the fibre manifold.

Our aim has been to develop a purely Hamiltonian approach and define a suitable bracket to help build a quantum theory. The only reliable way to convert a classical theory into a quantum theory is to define a suitable antisymmetric (or symmetric) bracket for observables of the theory which can be re-interpreted in quantum theory as a commutator (or anti-commutator). Our phase space has a very different character than the traditional phase space and our coordinate and momenta are differential forms of different degrees. In the traditional formalism the observables are real valued functions on the phase space and the definition of the Poisson bracket uses the pairing of one coordinate with one canonical momentum degree of freedom.

But that is special to one-time formalism of mechanics.

Whereas the Hamiltonian vector field for any observable exists for in mechanics, the same may not be so for fields. We have seen that the concept of a covariant bracket introduced by Peierls[7] in 1952 is a natural object to use in our Hamiltonian theory of fields. Here the rate of change of one quantity is taken when *the other quantity is added to the Hamiltonian as an infinitesimal perturbation* and vice-versa. The Poisson brackets of mechanics can be defined without reference to any Hamiltonian whereas the Peierls bracket requires the existence of a suitable governing Hamiltonian. Roughly speaking, the Poisson bracket can be described as the “equal time” Peierls bracket with zero Hamiltonian.

This gives us added insight into the Hamiltonian mechanics of one time formalism, particularly the concept of causality in systems with time dependent Hamiltonians [14]. The interesting features for one-time formalism of classical mechanics relating to causality and Peierls bracket which are revealed by our formalism of fields will be published elsewhere.

## A $i_Y\Theta$ for $Y = v^\mu\partial/\partial t^\mu$

As an illustration of the Noether theorem in our formalism let us evaluate  $i_Y\Theta$  for the present scalar field case for space-time translations. The vector field for constant infinitesimal displacement  $v^\mu$  is

$$Y = v^\mu \frac{\partial}{\partial t^\mu}$$

We are not assuming that space-time is flat or that  $Y$  are Killing fields of translation symmetry.

We know that

$$\begin{aligned} *p &= p_\mu * (dt^\mu) \\ &= \frac{1}{3!} \sqrt{-g} p_\mu g^{\mu\alpha} \varepsilon_{\alpha\nu\sigma\tau} (\nu\sigma\tau) \\ &\equiv \frac{1}{3!} \sqrt{-g} p^\alpha \varepsilon_{\alpha\nu\sigma\tau} (\nu\sigma\tau) \end{aligned}$$

where we introduce a convenient notation

$$(\nu\sigma\tau) \equiv dt^\nu \wedge dt^\sigma \wedge dt^\tau,$$



with similar notation for two or four factors of  $dt^\mu$  and we have defined the contravariant canonical momentum

$$p^\mu = g^{\mu\nu} p_\nu.$$

A simple calculation using

$$\begin{aligned} i_Y(dt^\mu \wedge dt^\nu \wedge dt^\sigma) &= v^\mu(dt^\nu \wedge dt^\sigma) - v^\nu(dt^\mu \wedge dt^\sigma) \\ &\quad + v^\sigma(dt^\mu \wedge dt^\nu) \end{aligned}$$

gives,

$$i_Y(*p) = \frac{1}{2!} \sqrt{-g} p^\alpha v^\beta \varepsilon_{\alpha\beta\sigma\tau} (\sigma\tau).$$

We can write this also as

$$i_Y(*p) = p_\mu v_\nu * (dt^\mu \wedge dt^\nu) = *(p \wedge Y^\flat)$$

where  $v_\mu = g_{\mu\nu} v^\nu$  and  $Y^\flat = v_\nu dt^\nu$  is the covariant field corresponding to  $Y$  after lowering the index by the metric.

As  $Y$  involves  $\partial/\partial t^\mu$  whose action on  $d\phi$  is zero

$$i_Y(*p \wedge d\phi) = (i_Y * p) \wedge d\phi,$$

and

$$\begin{aligned} i_Y[*p \wedge p] &= [i_Y * p] \wedge p - *p(i_Y p) \\ &= *(p \wedge Y^\flat) \wedge p - p(Y) * p. \end{aligned}$$

The formula  $i_Y(*p) = *(p \wedge Y^\flat)$ , although elegant, is not very useful for calculations. A straightforward expression for  $i_Y(*p) \wedge p$  is

$$i_Y(*p) \wedge p = [p_\mu(p.v) - v_\mu(p.p)] * (dt^\mu)$$

where

$$p.v = p_\mu v^\mu = \langle p, Y^\flat \rangle, \quad p.p = p_\mu p^\mu = \langle p, p \rangle.$$

Thus the calculation of  $i_Y \Theta$  proceeds as follows,

$$\begin{aligned} i_Y \Theta &= i_Y \left[ *p \wedge d\phi - \frac{1}{2} *p \wedge p - \frac{1}{2} m^2 \phi^2 * (1) \right] \\ &= (i_Y * p) \wedge \left( d\phi - \frac{1}{2} p \right) + \frac{1}{2} p(Y) * p \\ &\quad - \frac{1}{2} m^2 \phi^2 i_Y * (1) \end{aligned}$$

Evaluating it on the solution surface means we can put  $p = d\phi$ . Using expression for  $i_Y(*p) \wedge p$ ,  $p(Y) = p.v$  and the fact that

$$\begin{aligned} i_Y * (1) &= \sqrt{-g}(v^0[123] - v^1[023] + v^2[013] - v^3[012]) \\ &= v_\mu * (dt^\mu), \end{aligned}$$

we get

$$\begin{aligned} i_Y \Theta &= \left( \frac{1}{2}[p_\mu(p.v) - v_\mu(p.p)] + \frac{1}{2}(p.v)p_\mu \right) * (dt^\mu) \\ &\quad - \frac{1}{2}m^2\phi^2 v_\mu * (dt^\mu) \\ &= \left( p_\mu(p.v) - \frac{1}{2}[(p.p) + m^2\phi^2]v_\mu \right) * (dt^\mu) \Big|_0 \\ &= \langle d\phi, Y^\flat \rangle (*d\phi) - \frac{1}{2}[\langle d\phi, d\phi \rangle + m^2\phi^2](*Y^\flat) \end{aligned}$$

which can also be written in the useful form

$$i_Y \Theta = \left[ \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu} \left( g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + m^2\phi^2 \right) \right] v^\mu * (dt^\nu) \quad (18)$$

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